Adaptive numerical modeling using hierarchical Fup basis functions and control volume isogeometric anaylsis

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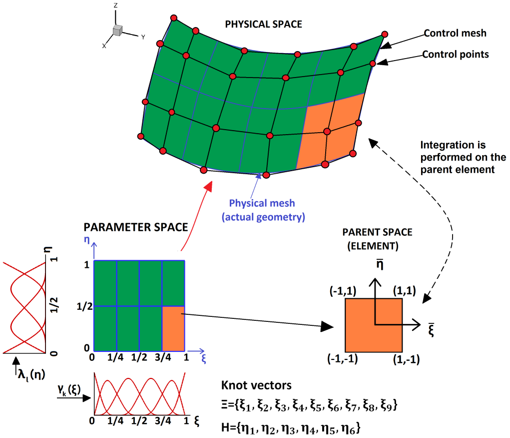
Isogeometric analysis

- A relatively new numerical approach - 2005. Hughes, Cottrell and Basilevs introduced the basics of **isogeometric anaylsis**
- Successfully applied in many areas of solid and fluid mechanics
- Significantly surpasses the classic FEM due to greater precision and increased continuity
- **Main idea: connect geometry modeling and numerical analysis procedure using the same type (spline) of basis functions**

T. J. Hughes, J. A. Cottrell, and Y. Bazilevs, "***Isogeometric analysis: Cad, finite elements, NURBS, exact geometry and mesh refinement***", Computer Methods in Applied Mechanics and Engineering, vol. 194, pp. 4135-4195, 10 2005.

J. A. Cottrell, T. J. R. Hughes, and Y. Bazilevs, "***Isogeometric Analysis Toward Intergration of CAD and FEA***", p. 335, 2009.

Isogeometric analysis



Transformation from the parameter (virtual) space to the physical (real) space:

$$x(\xi,\eta) = \sum_{j=1}^{k} x_j \phi_j(\xi,\eta)$$

$$y(\xi,\eta) = \sum_{j=1}^{k} y_j \phi_j(\xi,\eta)$$

The numerical solution in the virtual domain:

$$u(\xi,\eta) = \sum_{j=1}^{m} \alpha_j \varphi_j(\xi,\eta)$$

Isogeometric analysis

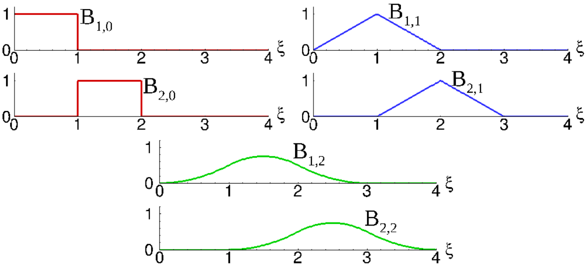
- Classical IGA uses:
  - B-spline, NURBS
  - Galerkin or collocation formulation

J. A. Cottrell, T. J. R. Hughes, and A. Reali, "***Studies of refinement and continuity in isogeometric structural analysis***", Computer Methods in Applied Mechanics and Engineering, vol. 196, pp. 4160-4183, sep 2007.
Y. W. Bekele, T. Kvamsdal, A. M. Kvarving, and S. Nordal, "***Adaptive isogeometric finite element analysis of steady-state groundwater flow***", International Journal for Numerical and Analytical Methods in Geomechanics, vol. 40, pp. 738-765, apr 2016.
G. Lorenzo, M. Scott, K. Tew, T. Hughes, and H. Gomez, "***Hierarchically refined and coarsened splines for moving interface problems, with particular application to phase-field models of prostate tumor growth***", Computer Methods in Applied Mechanics and Engineering, vol. 319, pp. 515-548, jun 2017.

A.-V. Vuong, C. Giannelli, B. Jüttler, and B. Simeon, "***A hierarchical approach to adaptive local refinement in isogeometric analysis***", Computer Methods in Applied Mechanics and Engineering, vol. 200, pp. 3554-3567, dec 2011.

Isogeometric analysis

- Classical IGA uses:
  - **B-spline**, NURBS
  - Galerkin or collocation formulation
- B-spline basis functions can be defined recursively:



for $n = 0$:

$$B_{i,0}(\xi) = \begin{cases} 1 & \xi_i \le \xi < \xi_{i+1} \\ 0 & otherwise \end{cases}$$

for $n > 0$:

$$B_{i,n}(\xi) = \frac{\xi - \xi_i}{\xi_{i+n} - \xi_i} B_{i,n-1}(\xi) + \frac{\xi_{i+n+1} - \xi}{\xi_{i+n+1} - \xi_{i+1}} B_{i+1,n-1}(\xi)$$

Basis functions

B-spline basis functions

- Piecewise polynomial function
- Compact support
- Non-negativity
- Partition of unity
- C^{n-1} continuity
- Numerical solutions are continuous
- Adaptive numerical procedures

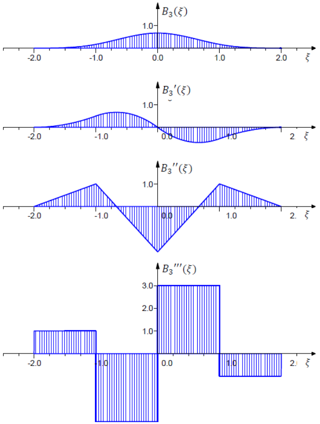


Figure: $B_3(\xi)$ spline basis function and its first three derivatives

Basis functions

B-spline basis function $\boldsymbol{B_n^l}$ defined on $\boldsymbol{\Xi^l}$ can be represented as a linear combination of $\boldsymbol{n+2}$ $\boldsymbol{B_n^{l+1}}$ basis functions defined on $\boldsymbol{\Xi^{l+1}}$ as:

$$B_{i,n}^l(\xi) = \sum_{k=0}^{n+1} c_{i,k}^n B_{2i+k,n}^{l+1}(\xi)$$

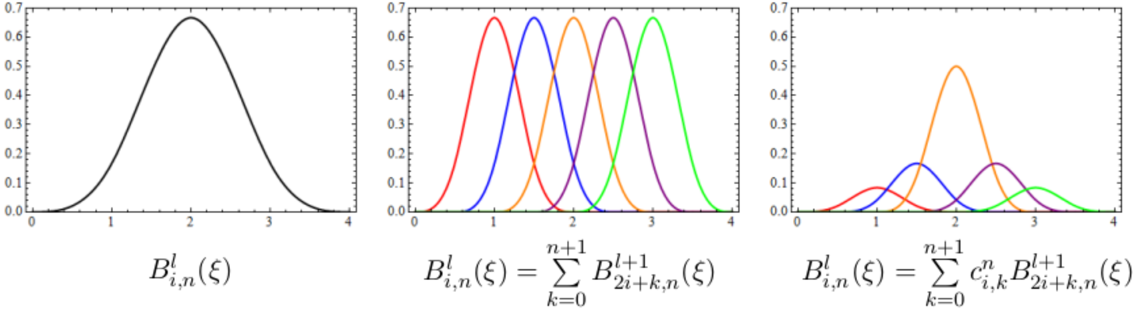


Figure: Linear combination of B_3 basis function on two consecutive levels (Wei *et al.* 2015.)

Basis functions

- Fup basis functions belong to the relatively lesser-known **atomic** or R_{bf} **basis functions**
- Rvachev and Rvachev 1971. are calling them atomic basis functions

V. L. Rvachev and V. A. Rvachev, "***On a finite function***", Dokl. Akad. Nauk Ukrainian SSR, ser. A, no. 6, pp. 705-707. 1971.

- Linear combination of the R_{bf} creates classical functions in mathematics
  - Atomic basis functions of the algebraic type: $up(\xi)$, $Fup_n(\xi)$
  - Atomic basis functions of the exponential type: $Eup(\xi)$, $EFup_n(\xi)$
  - Atomic basis functions of the trigonometric type: $Tup(\xi)$, $TFup_n(\xi)$
- R_{bf} basis functions - between classical polynomial functions and spline functions

Basis functions

Fup basis functions

- Infinitely differentiable splines (Gotovac and Kozulić 1999.)
- Compact support
- Non-negativity
- Partition of unity
- C^{∞} continuity
- Numerical solutions are continuous and smooth
- Adaptive numerical procedures

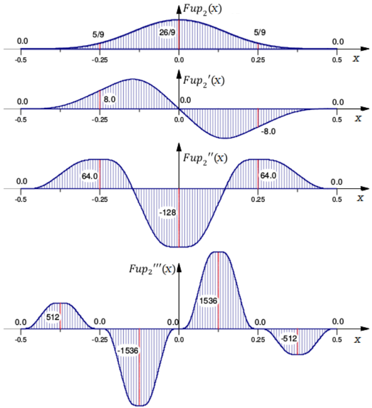


Figure: $Fup_2(x)$ basis function and its first three derivatives

Basis functions

Fup basis function $\boldsymbol{Fup_n^l}$ defined on Ξ^l can be represented as a linear combination of $\boldsymbol{n+2}$ $\boldsymbol{Fup_{n+1}^{l+1}}$ basis functions defined on Ξ^{l+1} as:

$$Fup_n^l(x) = \frac{1}{2^{n+1}} \sum_{k=0}^{n+1} C_{n+1}^k \cdot Fup_{n+1}^{l+1}\left(x - \frac{k}{2^{n+1}} + \frac{n+1}{2^{n+2}}\right)$$

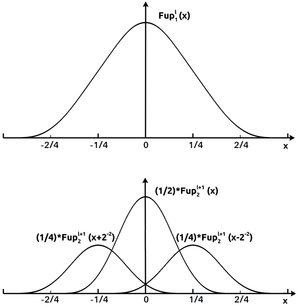


Figure: $Fup_1 \rightarrow Fup_2$

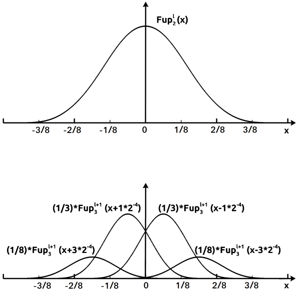


Figure: $Fup_2 \rightarrow Fup_3$

Basis functions

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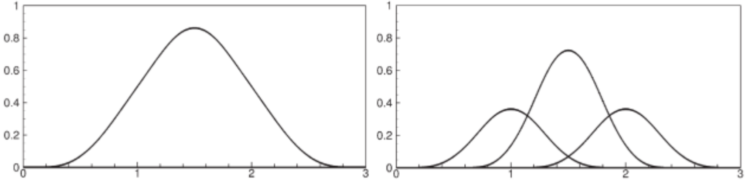

(A) $Fup_1^l(\xi)$ (B) $C_2^k Fup_2^{l+1}(\xi - \frac{k}{2^2} + \frac{2}{2^3});(k=0,1,2)$

Figure: Fup_1 basis function presented as linear combination of the Fup_2 basis functions (Kamber *et al.* 2020.)

Hierarchical Fup basis functions allow **local hp adaptation** $\rightarrow$ higher resolution levels have basis functions with **higher orders** and **smaller support** (higher frequencies).

Basis functions

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Idea: Creating a database of hierarchical Fup basis functions and implementation in adaptive numerical procedure in order to achieve

- Solution accuracy
- Solution stability
- Efficient procedure

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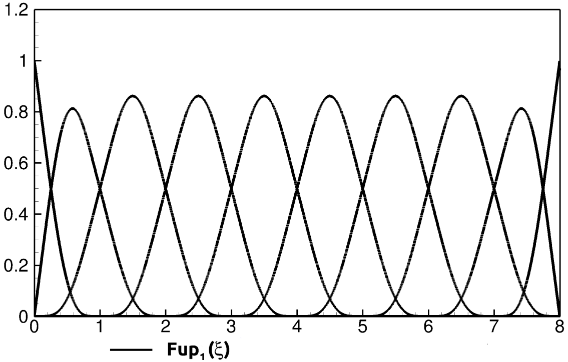
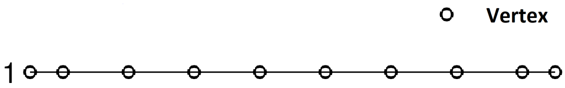
Methodology

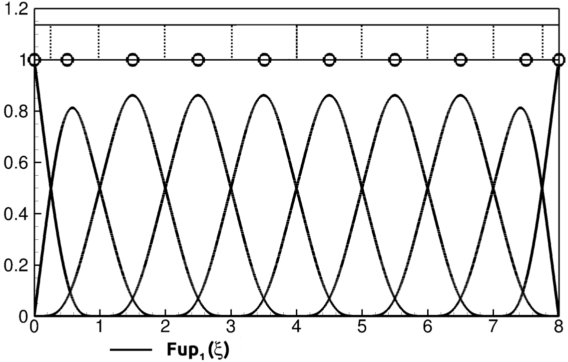
- The proposed model will be based on **Fup** basis functions and **control volume** formulation
- Due to certain similar properties with classical IGA, used method is called **control-volume isogeometric analysis** (**CV-IGA**)
- The main goal of CV-IGA is to use strong approximation and **adaptive** properties of the **hierarchical Fup** basis functions for numerical solutions of engineering problems arising in the field of structural mechanics and fluid mechanics with conservation properties of control volume formulation
- Control volume formulation ensures the conservation law locally and globally on the domain, and the stability of the numerical process with the computational costs that are between Galerkin (**high CPU time**) and collocation (**low CPU time**)

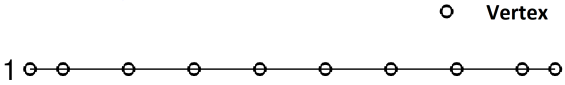
Methodology - 1D



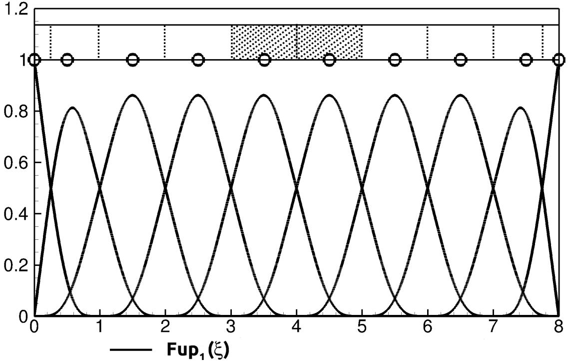


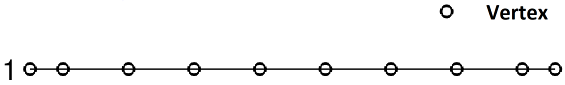
Methodology - 1D



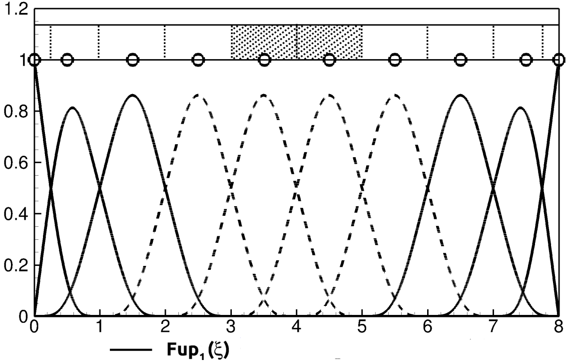


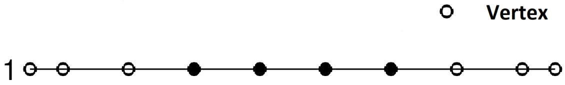
Methodology - 1D



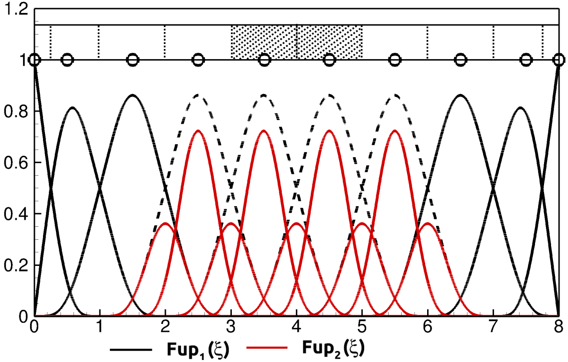


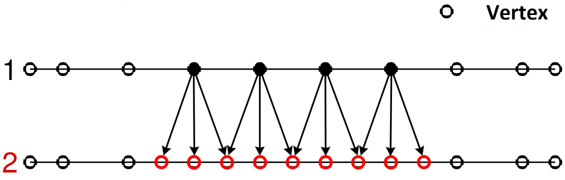
Methodology - 1D

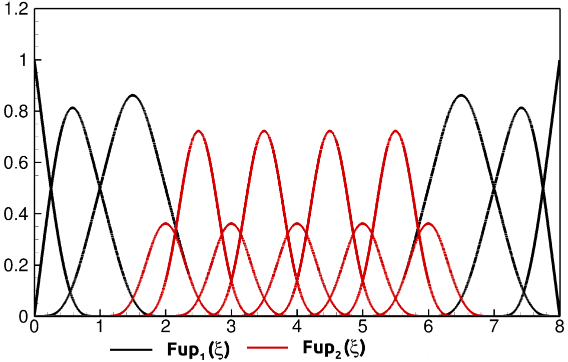




Methodology - 1D






1.2
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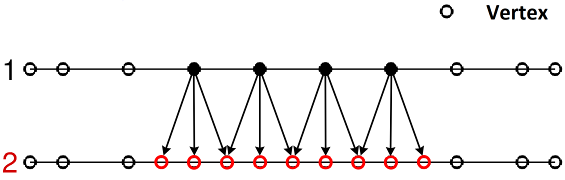
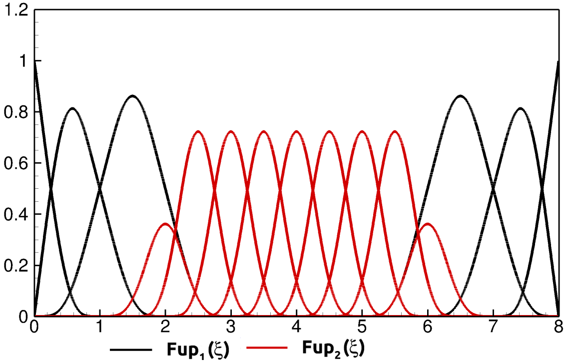
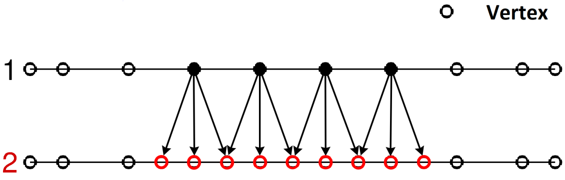
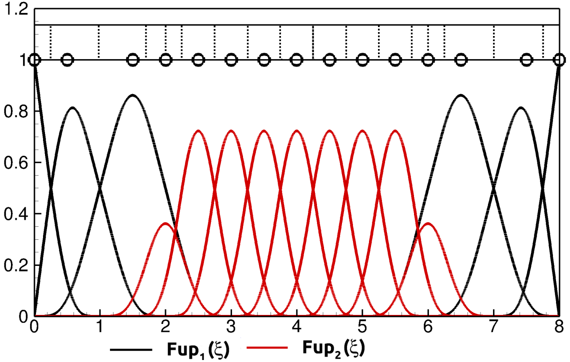
0.8
0.6
0.4
0.2
0
0
1
2
3
4
5
6
7
8
Fup$_1$(ξ)
Fup$_2$(ξ)


Vertex
1
2

Methodology - 1D






1.2
1
0.8
0.6
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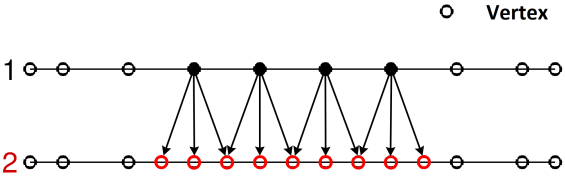
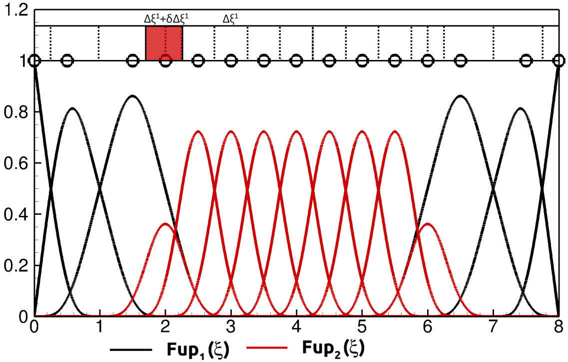
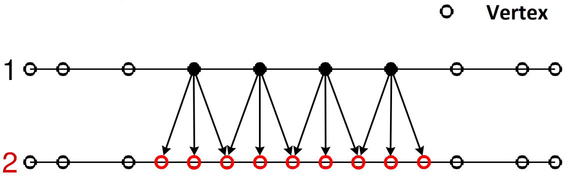
Fup1(ξ)
Fup2(ξ)

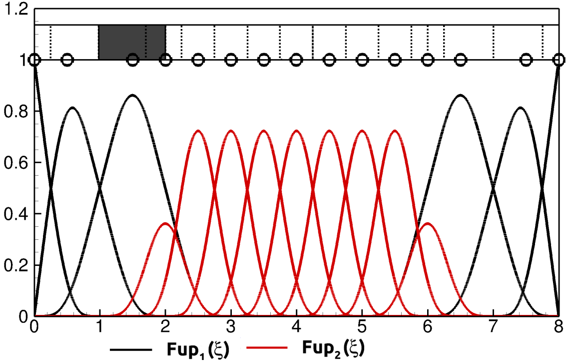

Vertex
1
2

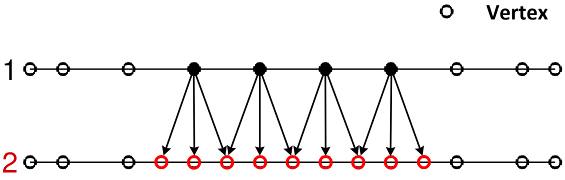
Methodology - 1D





Methodology - 1D





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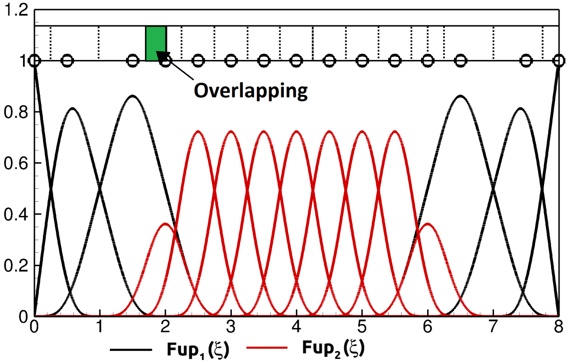
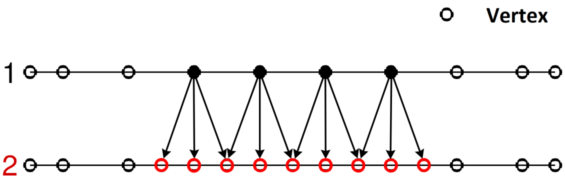
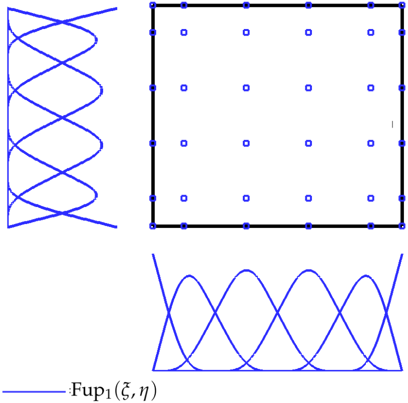




Methodology - 2D

$$Fup_n(\xi,\eta) = Fup_n(\xi) \cdot Fup_n(\eta)$$



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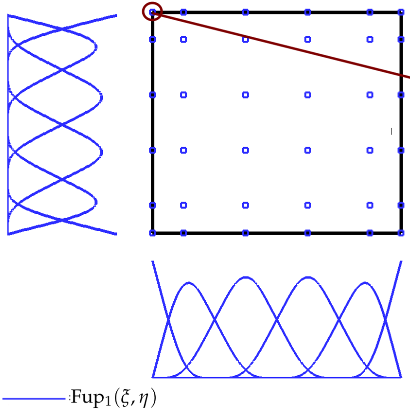

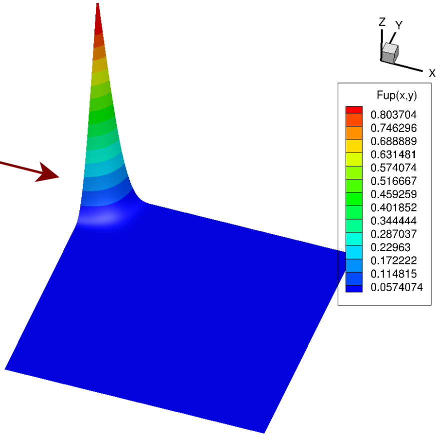

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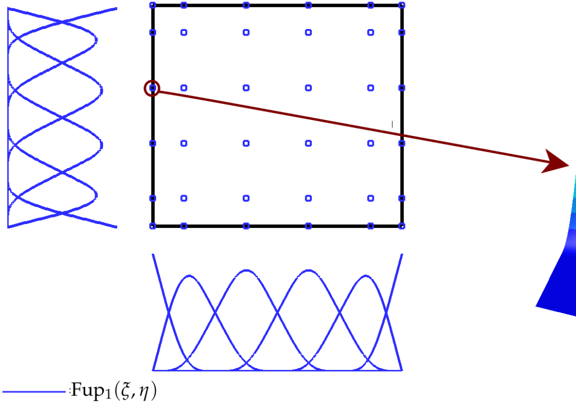

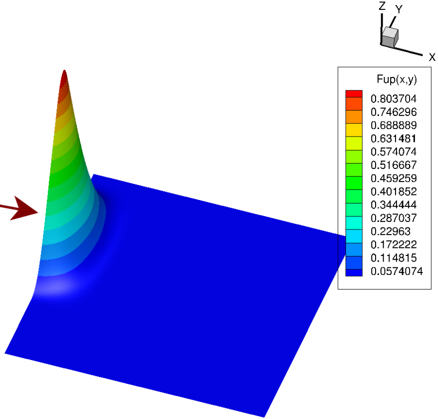

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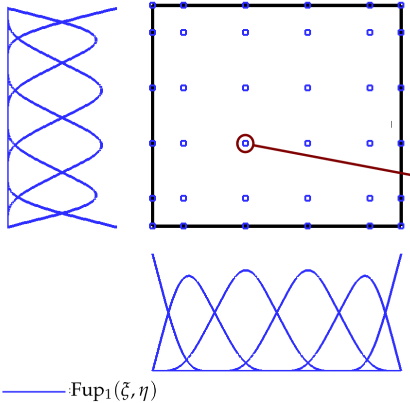
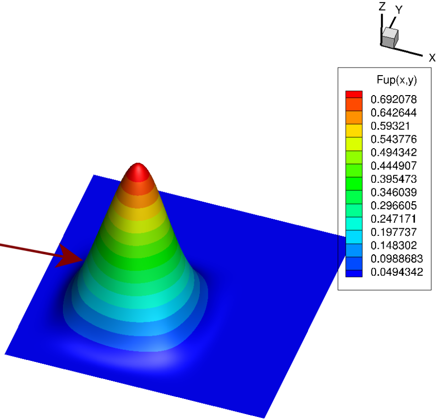
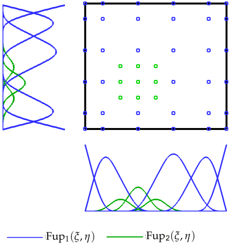
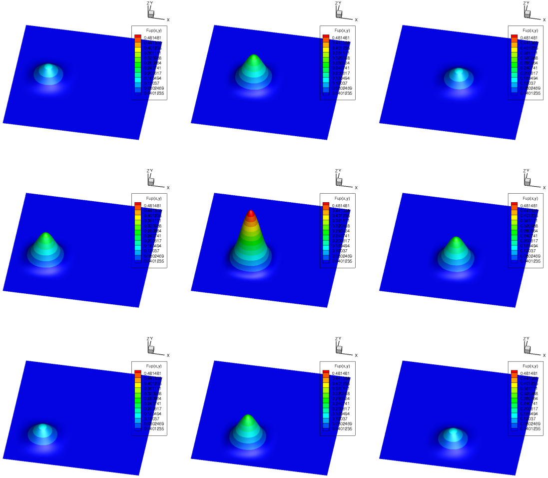
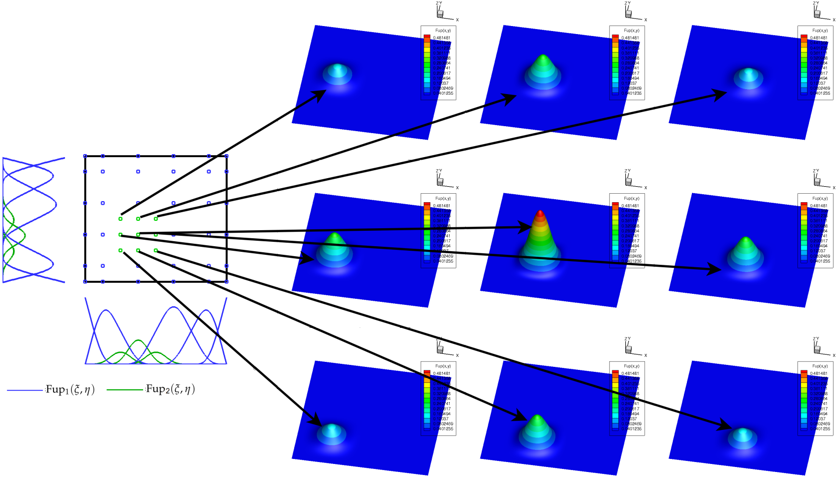


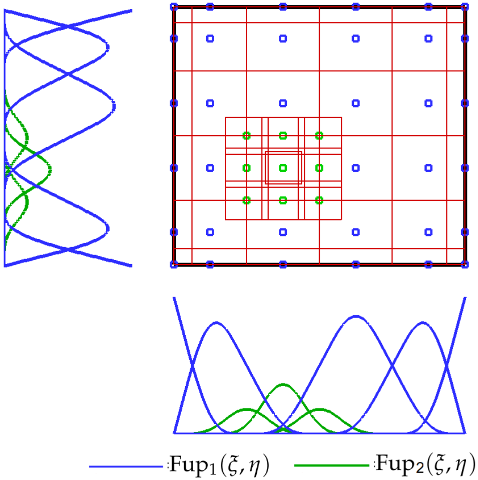



Fup1(ξ, η)
Fup2(ξ, η)


Fup(x,y)

Methodology - 2D




Fup$_1(\xi, \eta)$
Fup$_2(\xi, \eta)$

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Example - Wavefront well problem

Problem defined as:

$$\nabla \cdot (-\kappa \nabla u(x,y)) = f(x,y)$$

with boundary conditions:

$$u(x,y) = u_D(x,y)$$

Parameters:

$$\varepsilon_s = 10^{-4}$$

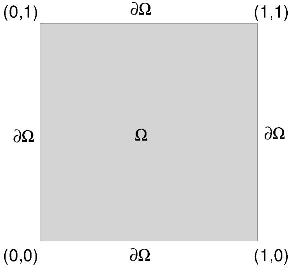
$$n^0 = 1$$

$$m_x^0 = 18;\ m_y^0 = 18$$

$$\Omega = [0,1]^2$$

$$x_c = y_c = -0.05$$

$$r_0 = 0.7;\ \alpha = 100$$



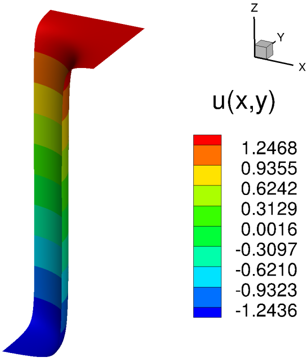
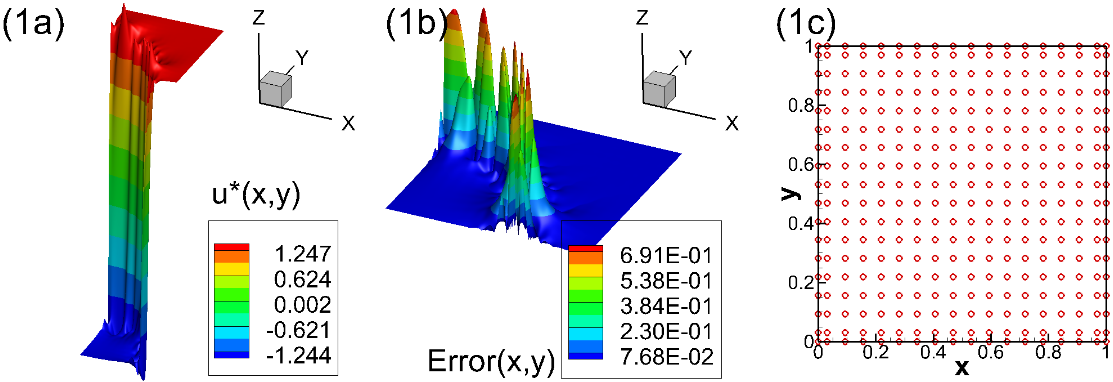
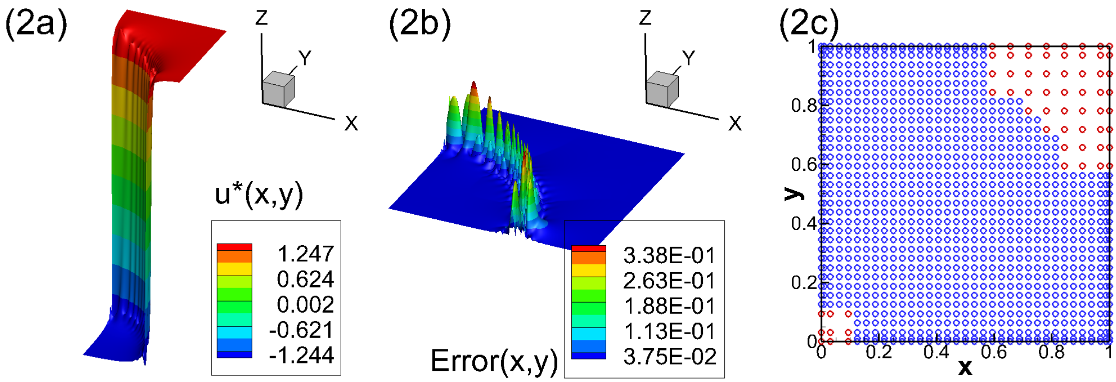


Figure: $u(x,y) = \arctan\left[\alpha\left(\sqrt{(x-x_c)^2 + (y-y_c)^2} - r_0\right)\right]$

$$\nabla \cdot (-\kappa \nabla u(x, y)) = f(x, y)$$

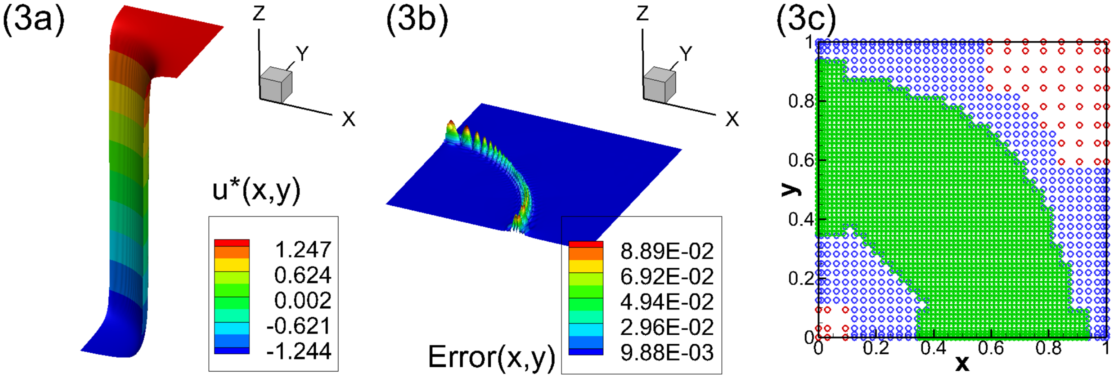


$$\nabla \cdot (-\kappa \nabla u(x, y)) = f(x, y)$$



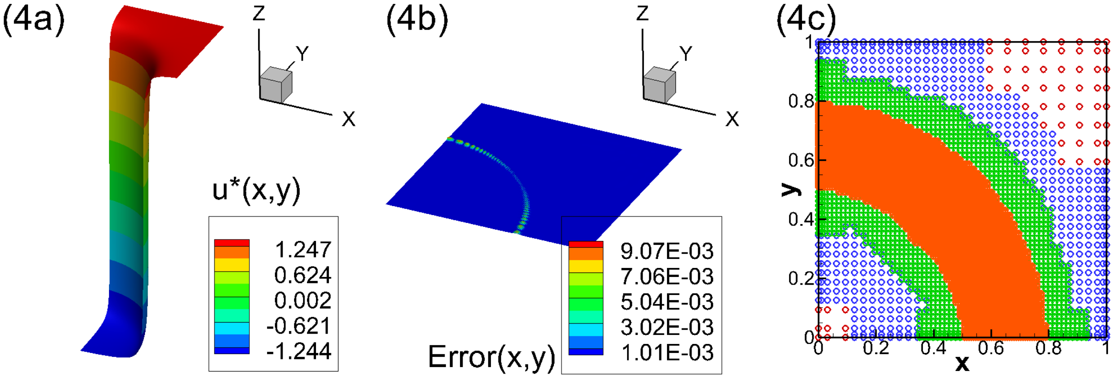
Wavefront well problem

$$\nabla \cdot (-\kappa \nabla u(x, y)) = f(x, y)$$



Wavefront well problem

$$\nabla \cdot (-\kappa \nabla u(x,y)) = f(x,y)$$



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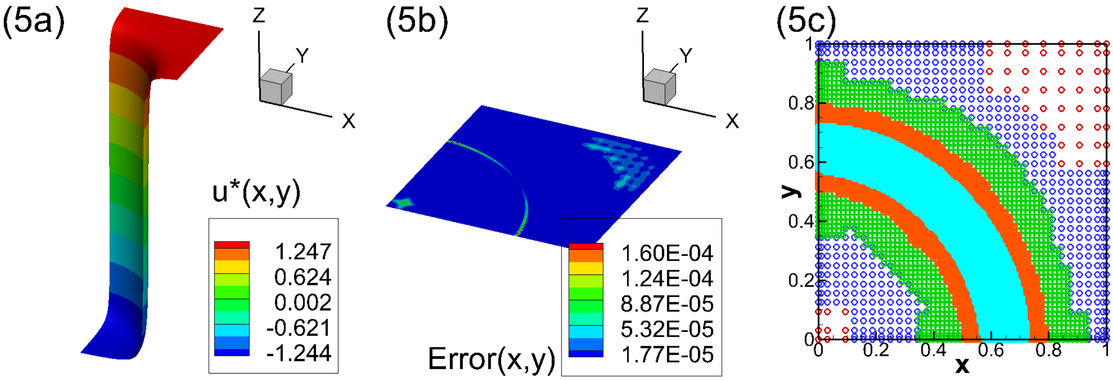

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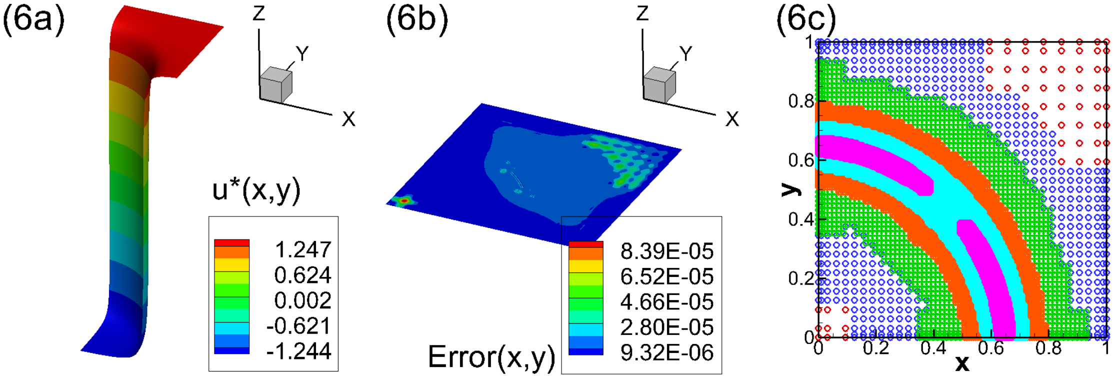
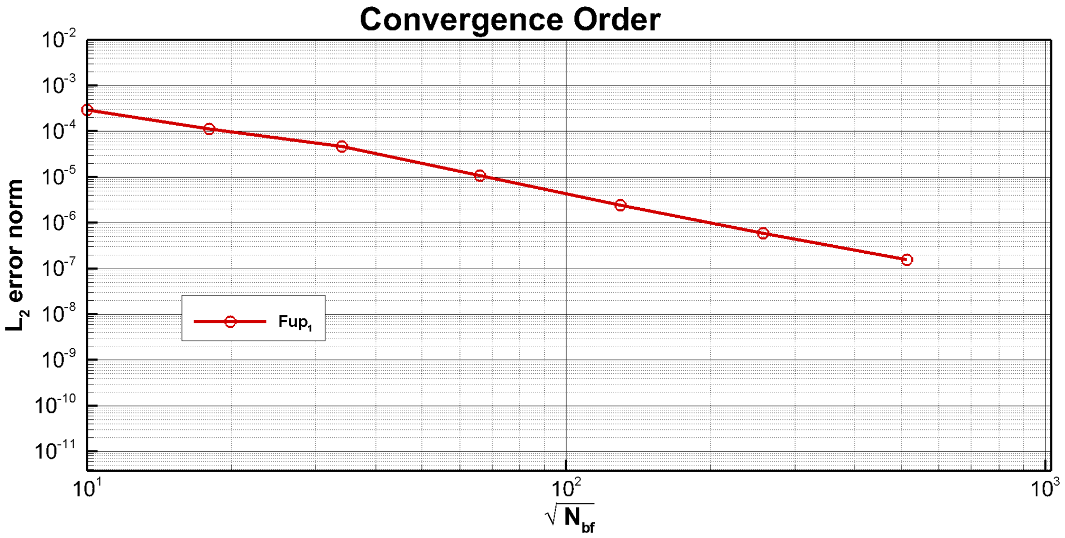
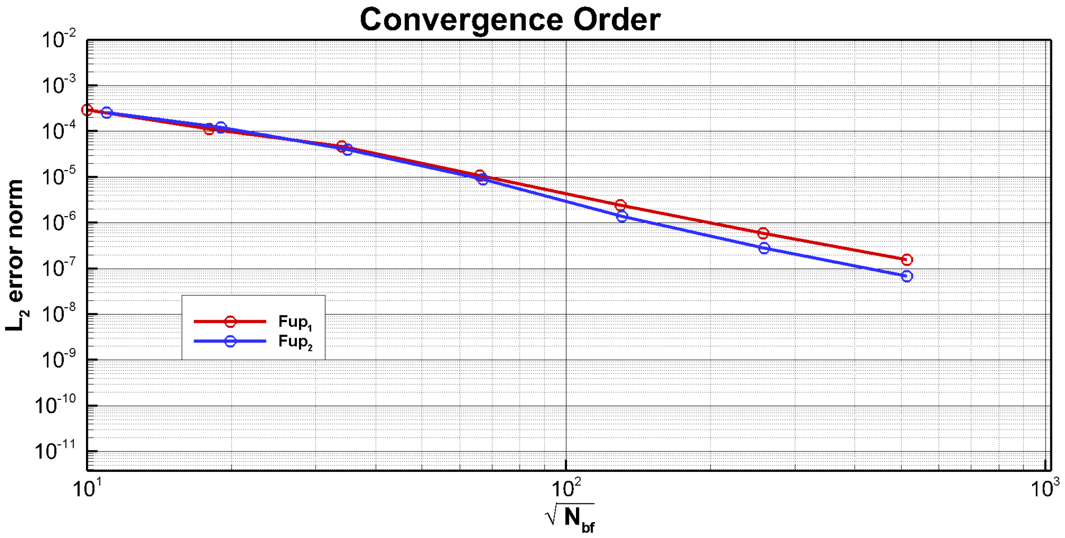


Wavefront well problem



Wavefront well problem



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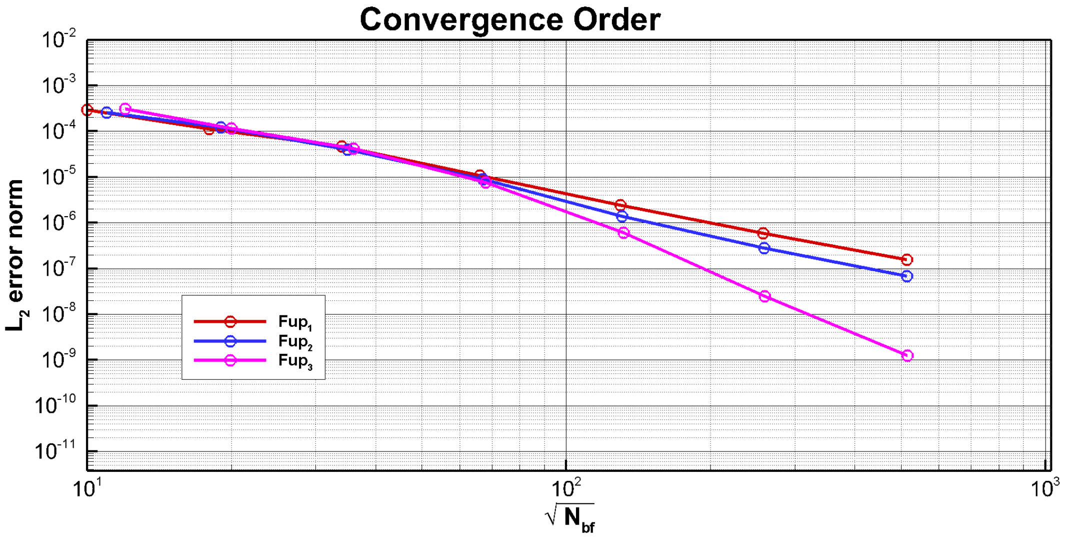
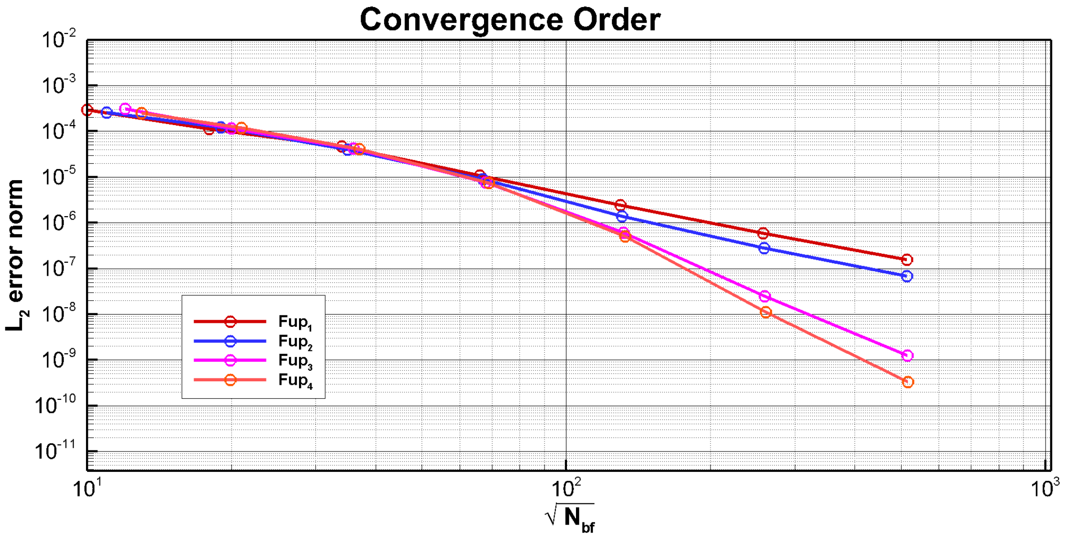
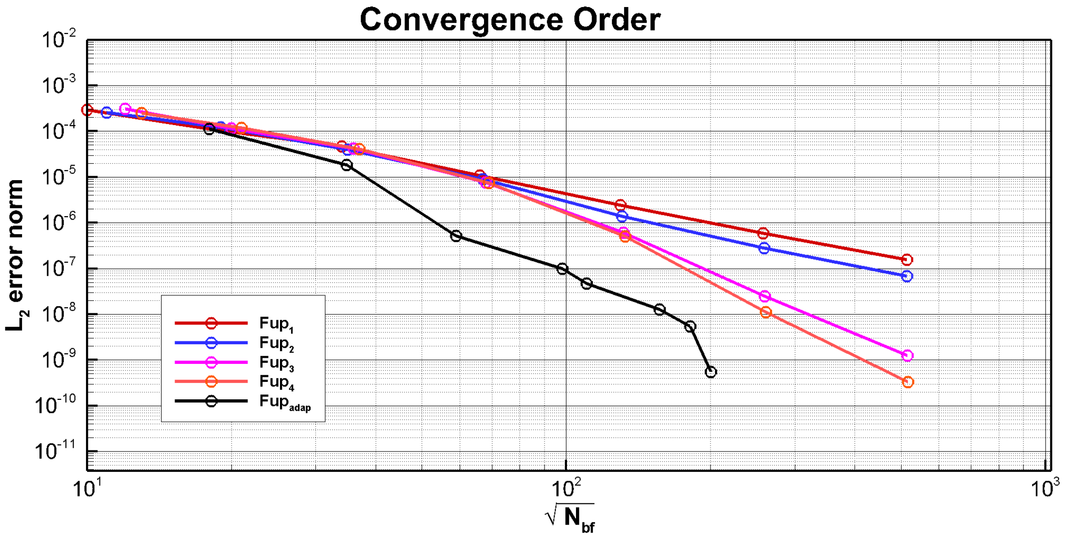


Wavefront well problem



Wavefront well problem



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Conclusions

hp-refinement

- One Fup_n can be replaced with $(n+2)$ Fup_{n+1} basis functions (one-dimensional case)
- This procedure enables adaptations, spectral convergence and increased efficiency
- B-splines do not support local hp-refinement in this form
- Powerful tool for adaptive numerical modeling
- Extend the proposed adaptive algorithm with Fup basis functions to the resolution of multiple-variable solution space-time scales arising in complex multiphysics problems

Further work

- Applications to 3D problems
  - Advection-dispersion problems
  - Heat transfers
  - Crack propagation

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Thank you for your attention