ADAPTIVE NUMERICAL MODELING USING HIERARCHICAL FUP BASIS FUNCTIONS AND CONTROL VOLUME ISOGEOMETRIC ANALYSIS

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1. Introduction

In this paper, a novel adaptive algorithm that is based on Fup basis functions [1, 2] that belong to the class of atomic functions and can be regarded as infinitely differentiable splines [3], is presented. Hierarchical Fup (HF) basis functions are closely related to the hierarchical B-splines (HB) and truncated hierarchical B-splines (THB). HF have the option of local *hp*-refinement such that they can replace certain basis functions at one resolution level with new basis functions at the next resolution level that have higher order (*p*-refinement) but also a smaller length of the compact support (*h*-refinement). This feature provides spectral convergence [4] and presents a substantial improvement in comparison to THB that enable polynomial convergence. HF basis functions are implemented into control volume IGA (CV-IGA) formulation. Poisson's equation is used for verification test.

2. Methodology

Fup basis functions belong to the class of atomic functions (see [1,3]) and span vector space of algebraic polynomials, while their properties are closely related to the more known B-splines.

Basis function Fup_n^l defined on a knot vector Ξ^l can be represented as a linear combination of $n+2$ Fup_{n+1}^{l+1} basis functions defined on Ξ^{l+1},

$$Fup_n^l(\xi) = \sum_{k=0}^{n+1} C_{n+1}^k \cdot Fup_{n+1}^{l+1}\left(\xi - \frac{k}{2^{n+1}} + \frac{n+1}{2^{n+2}}\right), \tag{1}$$

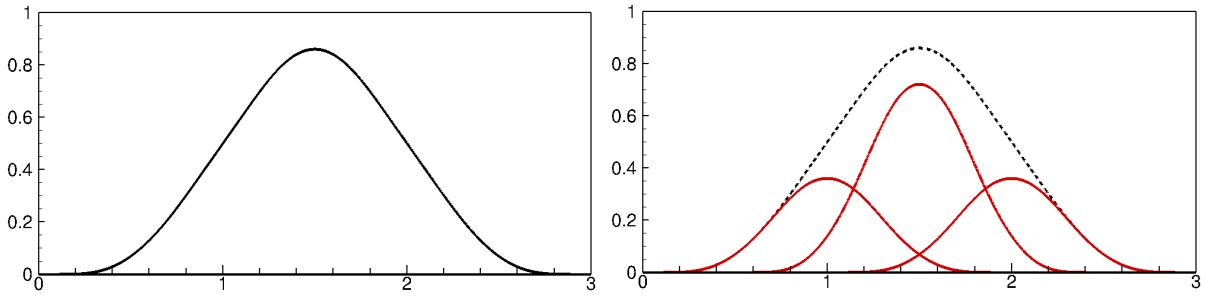
where n represents basis functions order and C_{n+1}^k are the refinement coefficients

$$C_{n+1}^k = \frac{1}{2^{n+1}}\binom{n+1}{k} \tag{2}$$

The $n+2$ basis functions Fup_{n+1}^{l+1} are called the *children* of Fup_n^l, denoted as

$$chdFup_n^l(\xi) = \left\{ Fup_{n+1}^{l+1}\left(\xi - \frac{k}{2^{n+1}} + \frac{n+1}{2^{n+2}}\right) \middle| k = 0,1,...,n+1 \right\} \tag{3}$$

Figure 1a shows a basis function Fup_1^l defined on a knot vector $\Xi^l = \{0,1,2,3\}$, and Figure 1b shows its three children $\mathrm{Fup}_2^{l+1}\left(\xi - \frac{k}{2^2} + \frac{2}{2^3}\right)$ defined on a knot vector $\Xi^{l+1} = \left\{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3\right\}$, where $k =$



(a) $\mathrm{Fup}_1^l(\xi)$ (b) $C_2^k \mathrm{Fup}_2^{l+1}\left(\xi - \frac{k}{2^2} + \frac{2}{2^3}\right); (k = 0,1,2)$

Figure 1: Refinability of a $\mathrm{Fup}_1(\xi)$ basis function. (a) $\mathrm{Fup}_1^l(\xi)$ is defined on the knot vector $\Xi^l = \{0,1,2,3\}$; and (b) $\mathrm{Fup}_2^{l+1}\left(\xi - \frac{k}{2^2} + \frac{2}{2^3}\right)$ are defined on a knot vector $\Xi^{l+1} = \left\{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3\right\}$ with $C_2^k = \frac{1}{2^2}\binom{2}{k}$ for $k = 0,1,2$.

$0,1,2$. According to Eq. (1), Fup_1^l (Figure 1b - black dashed curve) can be represented by a weighted summation of its three children Fup_2^{l+1} (Figure 1b - red solid curve). In contrast to THB, hierarchical Fup basis functions (HF) enable *hp*-adaptive methods because each next resolution level not only decreases compact support but also increases the order of the basis functions (*hp*-refinement). Multidimensional Fup basis functions are obtained as tensor products of the one-dimensional basis functions defined for each coordinate direction.

To derive CV-IGA, the 2D parameter space defined by uniform (open) knot vector is shown on Figure 2. Domain is subdivided into a set of CVs such that each CV surrounds one corresponding vertex (Figure 2 - black circles) of basis functions and CVs boundary is defined at half distance between neighboring vertices.

In Figure 2, the focus is on the grid point $V_{i,j}$ (represents vertex), which has the grid points V_l, V_r, V_u and V_d as its neighbors and $CV_{i,j}$ boundaries (Figure 2 - red line) are marked as $\Gamma_{i,j,l}, \Gamma_{i,j,r}, \Gamma_{i,j,d}, \Gamma_{i,j,u}$.

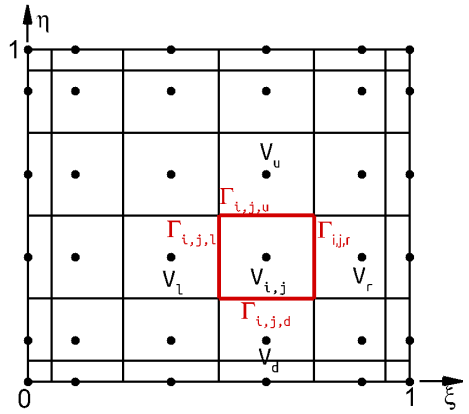


Figure 2: Control volume scheme for two-dimensional case.

Figure 3 shows a nested sequence of CVs domain, together with the corresponding vertices for each resolution level l, where each CV is linked with only one vertices i.e., the number of basis functions corresponds to the number of CVs.

CVs on the higher level $(l \geq 1)$ are build in a slightly different way then on starting level where all CVs are nonoverlapping. Each CVs boundary on the higher level (see Figure 3) are positioned exactly half the length of the characteristic intervals $\Delta\xi$, $\Delta\eta$ from the corresponding Fup basis function vertex, thus higher levels (CVs) are overlapping with lower levels (CVs). Overlapping process makes this algorithm even more robust, but at the cost of more expensive numerical integration.

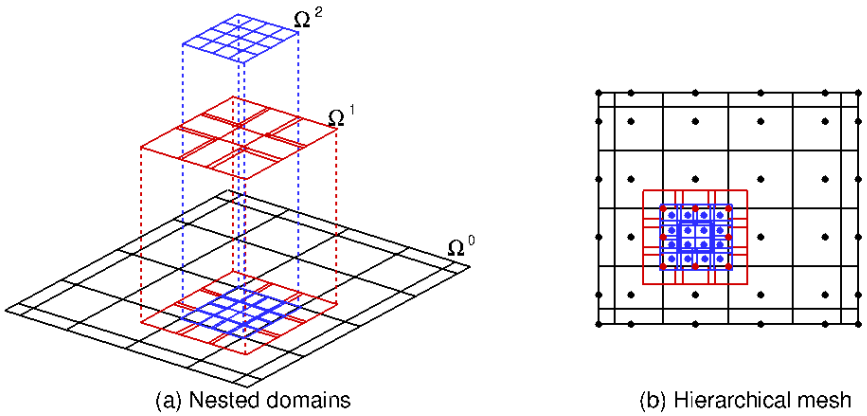


Figure 3: A nested sequence of CV domains for the construction of the Fup hierarchy according to relation $\Omega^l \supseteq \Omega^{l+1}$ for $l = 0, 1, 2$ for two-dimensional case.

3. Numerical example

Poissson equation, so called wavefront well problem is considered for verification test. It is commonly used example for testing adaptive refinement algorithms because of a steep wave front in the interior of the domain [5–7]. Parameters determine the steepness and location of the wave front.

Problem is defined in the form

$$\nabla \cdot (-\kappa \nabla u(x,y)) = f(x,y) \quad (x,y) \in \Omega \tag{4}$$

with boundary conditions

$$u(x,y) = u_D(x,y) \quad (x,y) \in \partial\Omega \tag{5}$$

The numerical simulation domain is defined by a square area $\Omega = [0,1]x[0,1]$ where the boundaries are $\Gamma_D = \partial\Omega$ and $\Gamma_N = \emptyset$ (see Figure 4a). The exact analytical solution for the pressure field is given by:

$$u(x,y) = \arctan(\alpha(r - r_0)) \quad \text{where} \quad r = \sqrt{(x-x_c)^2 + (y-y_c)^2} \tag{6}$$

where x_c and y_c represents center of the circular wave front, r_0 is the distance from the wave front to the center of the circle, and α gives the steepness of the wave front.

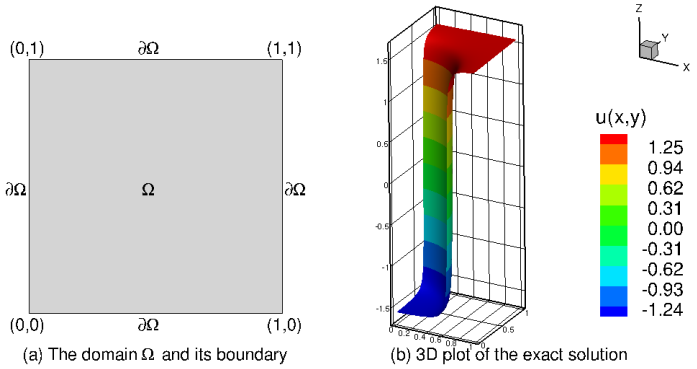


Figure 4: Numerical domain and exact solution plot of the wave well problem defined by Eq. (4)

Figure 5 presents the numerical solution of the wave front given by Eq. (4). Numerical solution becomes closer to the real solution (see Figure 5 1b-6b) with every new level. Adaptive criteria is used to check conservation error for each quarters of the particular i-th CV on the current resolution level. Quarters of the CVs are used because CV formulation exactly satisfies governing equation (i.e., the weak integral form of the conservation law), over each CV on the current resolution level. Adaptive procedure is repeated until conservation error is less then the prescribed threshold at each quarter of the CVs. The

adaptive grid captures the front (see Figure 5 1c-6c). For given parameters $x_c = y_c = -0.05$, $r_0 = 0.7$ and $\alpha = 100$ and using HF, six levels are needed in order to find numerical solution that has conservation error less then prescribed error threshold $(\varepsilon_s = 1 \cdot 10^{-4})$ on all quarters of the CVs.

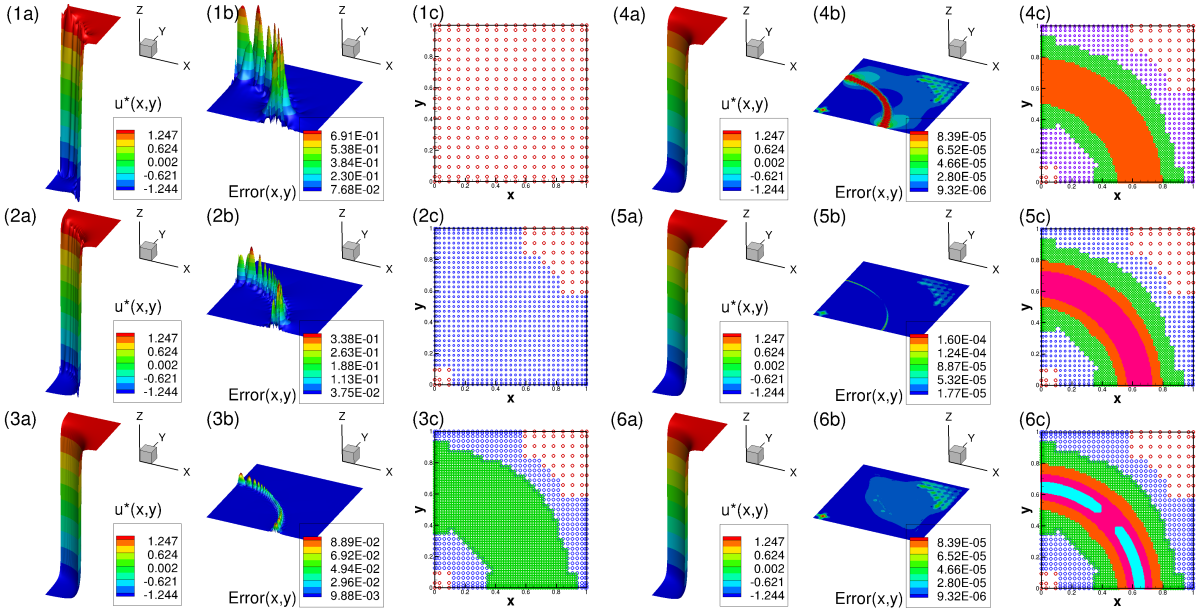


Figure 5: Numerical solution of the wave front well problem defined by Eq. (4). (a) HF approximation, (b) the absolute difference between the numerical and exact solution and (c) the adaptive grid on different resolution levels where each color represents Fup basis function vertices on different level.

4. Conclusions

A novel adaptive algorithm based on hierarchical Fup (HF) basis functions and the control volume IGA formulation is presented. Fup basis functions (infinitely differentiable splines) belong to the class of atomic functions and have the option of local *hp*-refinement. The application of hierarchical Fup basis functions enables higher continuity and smoothness of the solution, and at the same time, provides spectral convergence, while control volume formulation retains the conservation property of governing equations.

References

[1] V. L. Rvachev and V. A. Rvachev. On a finite function. *Dokl. Akad. Nauk Ukrainian SSR, ser. A*, (6):705–707., 1971.

[2] Blaz Gotovac. *Numerical modelling of engineering problems by smooth finite functions (In Croatian)*. PhD thesis, 1986.

[3] B. Gotovac and V. Kozulic. On a selection of basis functions in numerical analyses of engineering problems. *International Journal for Engineering Modelling*, 12(1-4):25–41, 1999.

[4] G. Kamber, H. Gotovac, V. Kozulic, L. Malenica, and B. Gotovac. Adaptive numerical modeling using the hierarchical fup basis functions and control volume isogeometric analysis. *International Journal for Numerical Methods in Fluids*, 92(10):1437–1461, 2020.

[5] W. Rachowicz, J.T. Oden, and L. Demkowicz. Toward a universal h-p adaptive finite element strategy part 3. design of h-p meshes. *Computer Methods in Applied Mechanics and Engineering*, 77(1-2):181–212, December 1989.

[6] J.T. Oden and Abani Patra. A parallel adaptive strategy for hp finite element computations. *Computer Methods in Applied Mechanics and Engineering*, 121(1):449 – 470, 1995.

[7] William F. Mitchell. A collection of 2d elliptic problems for testing adaptive grid refinement algorithms. *Applied Mathematics and Computation*, 220:350 – 364, 2013.